The Red Queen visits Minkowski Space

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arXiv:physics/0610190 v1 23 Oct 2006

Abstract

When Alice went *Through the Looking Glass* [1], she found herself in a situation where she had to run as fast as she could in order to stay still. In accordance with the dictum that truth is stranger than fiction, we will see that it is possible to find a situation in special relativity where running towards one's target is actually counter-productive. Although the situation is easily analysed algebraically, the qualitative properties of the analysis are greatly illuminated by the use of space-time diagrams.

Although tachyons (particles which travel faster than light) are not at present observed experimentally, they arise naturally in superstring theory, where their consequences require investigation: one example of such an inquiry is found in [2]. Outside this context, tachyons have also been considered from advanced viewpoints, as in [3], in which it was found that the obvious problems associated with causality might be illusory; and from elementary viewpoints, as in [4] where simple geometrical properties of a tachyonic wavefront were considered.

This article takes a brief look at how tachyons appear to move from the point of view of various inertial observers in special relativity. The results are reminiscent of Alice's experience through the looking glass, where she had to run as fast as she could just to stay still. Here we will find that the situation can be worse even than that: it is possible for a target to recede faster, the faster you chase it.

Although the results are easy to obtain algebraically, it is the use of space-time diagrams that renders the situation intelligible. Finally, the relative strengths of the algebraic and diagrammatic approaches are briefly discussed. The article is presented in a discursive manner, and should be accessible to students who have taken a course in special relativity.

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We will restrict our attention to situations in which all motion takes place along the x-axis in Minkowski space, so that we can consider kinematics in a two-dimensional space-time; furthermore, we will suppose that units have been chosen so that $c = 1$. So, consider an inertial frame Σ, with associated coordinates (t, x). In this space-time we have three observers, A, B and C. A is at rest at $x = 0$, B is heading in the positive x-direction with speed $1/2$ and C is travelling in the negative x-direction at speed $1/2$; they all meet at $x = 0$ when $t = 0$.

Now consider the following announcements, made just after the three observers pass each other:

A   I just saw something travelling in the positive x-direction at speed $u_A = 3/2$
B   I just saw something travelling in the positive x-direction at speed $u_B = 4$
C   I just saw something travelling in the positive x-direction at speed $u_C = 8/7$

The surprising thing is that these three comments should all apply to observations of the same object. The reason it is surprising is that since B is travelling in the positive x-direction, and C in the negative x-direction, we would normally expect that B should see an obect travelling in the positive x-direction travel slower than A, while C would see it travel faster. But, contrariwise, the observations have $u_C < u_B < u_A$.

In order to resolve this apparent paradox, let us consider how velocities transform between frames of reference in special relativity. We will see that this is, in fact, independent of the velocity to be transformed, and so the usual relativistic 'addition of velocities' is valid even when we are working with a tachyonic particle.

For simplicity, we consider only one dimension of space. So let Σ, with coordinates (t, x), be some nominal rest frame, and let Σ' be a frame whose origin is travelling with speed V in the positive x-direction in Σ, with coordinates (t', x'). Suppose also that the event with $t = 0, x = 0$ also has $t' = 0, x' = 0$. Then the coordinates (t, x) and (t', x') are related by the usual Lorentz transformation

$$\begin{aligned} t &= \gamma(t' + Vx') \\ x &= \gamma(x' + Vt') \end{aligned} \tag{1}$$

where $\gamma = 1/\sqrt{1 - V^2}$.

Now, suppose we have an object whose world-line is given in terms of (t, x) by $x = ut$, so that it is travelling with speed u in the positive x-direction. Expressing x and t in terms of x' and t' we immediately obtain

$$x' + Vt' = u(t' + Vx')$$

which is easily rearranged to give

$$x' = \frac{u - V}{1 - uV} t'.$$

Denoting by u' the speed in the positive x'-direction, as measured in Σ', we have

$$u' = \frac{u - V}{1 - uV}.$$

Thus we have the usual 'addition of velocities' rule, and observe that this result is quite independent of the sign or size of u.

So we can now easily check that if A sees an object moving to the right at speed $3/2$, then B (for whom $V = 1/2$) will attribute to it a speed of

$$\frac{3/2 - 1/2}{1 - 3/4} = 4$$

while C (with $V = -1/2$) will find its speed to be

$$\frac{3/2 + 1/2}{1 + 3/4} = 8/7.$$

We can see, then, from the algebraic properties of the Lorentz transformations, that this is indeed how the velocity of a tachyonic particle would transform between frames of reference. In fact, the speed of the tachyon as measured by a moving observer has still more peculiar properties.

First, let us look at the tachyon's velocity in a frame moving with velocity V; if we call this velocity u', then we saw above that

$$u' = \frac{u - V}{1 - uV}.$$

Differentiating this with respect to V, we obtain

$$\frac{\partial u'}{\partial V} = \frac{u^2 - 1}{(1 - uV)^2}$$

which is always positive; hence, the faster you chase a tachyon, the faster it recedes.

However, even this is not as straighforward as it looks at first glance. Examining the form of u' more carefully, we make the following observations:

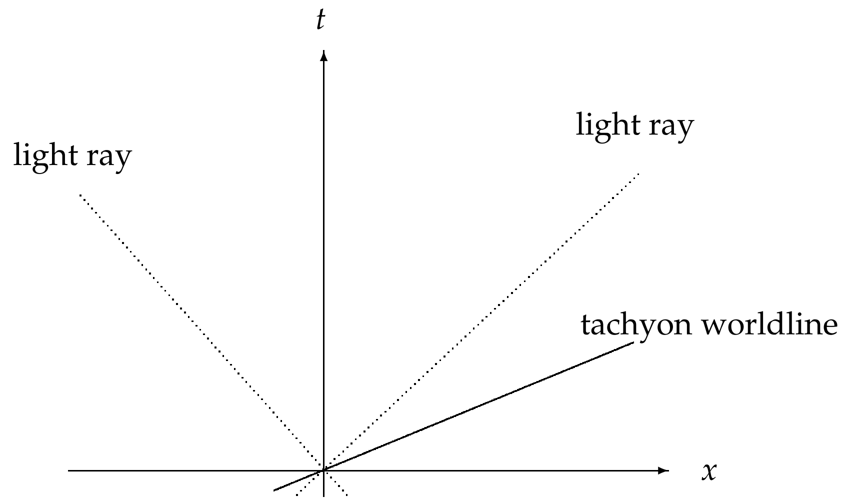
1. As $V \to -1$, $u' \to 1$
2. For V between -1 and $1/u$, u' is increasing, and $u' \to \infty$ as $V \to 1/u$ from below.
3. As $V \to 1/u$ from above, $u' \to -\infty$.
4. u' is increasing as V increases from $1/u$ to 1, and as $V \to 1$, $u' \to 1$.

So we see that the tachyon is seen to travel faster than light by all inertial observers; but that as the speed of the moving observer increases, the speed with which the tachyon recedes increases without bound until suddenly it switches from receding with extremely high speed to approaching with extremely high speed, but then the speed of approach decreases as the speed of the moving observer continues to increase.

Again, although it is simple to derive all this by applying simple algebra to the velocity transformation rule, it is unclear what is really going on here.

In order to obtain some insight into the situation, we consider some space-time diagrams [5]. (It is worth noting that this form of space-time diagram is not the only one; Shadowitz [6] considers a variety of space-time diagrams, each of which has its strengths. However, we will make use of only the form due to Minkowski, and leave investigation of the others to the reader.)

First, consider a space-time diagram that shows only the rest-frame of A, namely Σ, and the tachyon worldline in it. As is customary, the units of distance and time are chosen such that light rays are at 45° to the vertical.

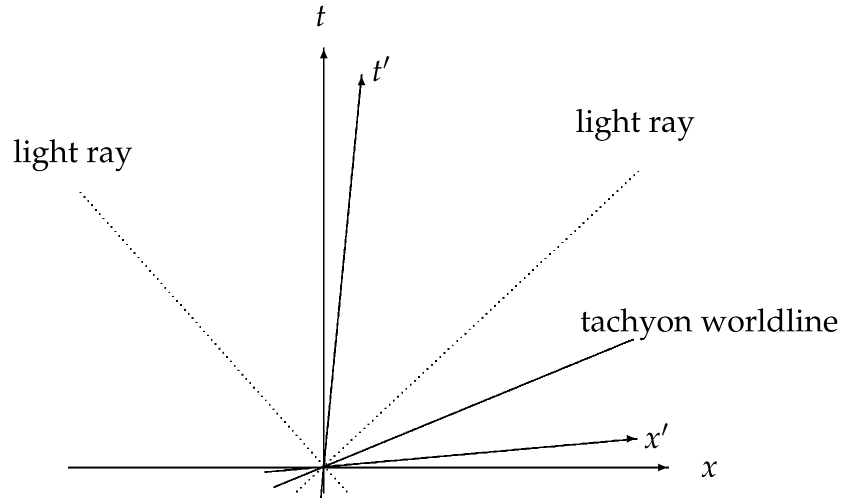


So we clearly see that the tachyon is proceeding in the direction of increasing x faster than a light ray in frame Σ.

We can now introduce to the diagram the t' and x' axes of the rest frame of an observer moving at constant velocity. First, we note that the relation given in equation 1 can be rearranged to give

$$t' = \gamma(t - Vx)$$
$$x' = \gamma(x - Vt)$$

so that the x'-axis is given by $t' = 0$, i.e. $t = Vx$, and the t'-axis by $x' = 0$, i.e. $t = x/V$.



By inspecting this diagram we see that the situation is not so counter-intuitive after all. In the same way as a Lorentz transformation to a frame with positive speed in the x-direction will make the worldline of a particle travelling in that direction with a lesser speed 'more timelike' in the sense that it becomes nearer the t-axis, such a Lorentz transformation will make the worldline of a tachyon 'more spacelike'; so that an observer who is travelling in the same direction as a tachyon attributes to it a greater speed than the stationary observer. Furthermore, as the speed increases, it reaches a value at which the tachyon worldline is a line of simultaneity (the perceived speed taking on unboundedly large values); and for larger speeds, the tachyon worldline has speed with unboundedly large value to the left, which then reduce in magnitude as the speed of the observer continues to grow.

The diagrammatic investigation also brings out the symmetry between this situation and that of the different velocities ascribed by observers to an object moving with constant sub-luminal speed; for just as by chasing sufficiently fast an observer can make this object's worldline will pass through the his line of constant position, so he can make the tachyon's worldline pass through his line of constant time.

From this investigation, then, we can see the respective strengths of the algebraic and diagrammatic approaches to analysing this situation. The algebraic approach provides complete quantitative information, but does little to give any insight into the qualitative behaviour of the transformed velocities. On the other hand, spacetime diagrams make the qualititative behaviour comprehensible, without giving easy access to the numerical values of observed speed.

References

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